AN INTRODUCTION TO DIFFEOMORPHISM GROUPS

DEREK R. AOKI

Abstract. In this paper, we introduce a reader with knowledge of only basic group theory and relatively little of differential geometry to the subject of diffeomorphism groups via the proof of Kathryn Mann's that $\mathrm{Diff}_c(M)$ is perfect.

1. Introduction

Studying the symmetries of an object is well-known to be an important way to study the object itself, and the objects of differential geometry are no exception to this. We aim to provide an invitation to the study of the structure of groups of diffeomorphisms of manifolds, and we do this by providing an exposition of Kathryn Mann's proof of the perfectness of $\mathrm{Diff}_c(M)$ [14]. While the aforementioned proof is elementary to those already familiar with the study of diffeomorphism groups, it takes for granted a variety of topics from Lie theory, differential geometry, and more. We make the proof accessible to a far wider audience, introducing precisely the necessary information for a student with only basic group theory and the differential geometry of the first three chapters of [13]. We choose this result in particular because it is a landmark by Thurston in the area [19], and this proof in particular because it naturally brings up a variety of important concepts and results in the area, such as the topology of diffeomorphism groups, isotopies, Lie groups, exponentials of vector fields, the Thurston tricks, and the perfectness and simplicity of certain important diffeomorphism groups.

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In Section 1, we introduce diffeomorphism, topological, and Lie groups, as well as our main theorem that $\mathrm{Diff}_c(M)$ is perfect. In Section 3, we introduce partitions of unity and prove their existence, prove the Fragmentation Lemma, and reduce to the case $M = \mathbb{R}^n$. In Section 4, we make a detour to diffeomorphisms of the circle, covering $\mathrm{PSL}(2, \mathbb{R})$ and invoking Herman's theorem. In Section 5, we induct on n to prove the theorem for $n \geq 2$. In Section 6, we conclude with an overview of the history of the problem and some possibilities for further reading.

2. Diffeomorphism Groups

We will throughout this paper concern ourselves only with smooth manifolds, which we consider to be Hausdorff and second-countable. Also, our convention will be that a diffeomorphism is smooth with a smooth inverse. We now describe rigorously our central object of study, the diffeomorphism group of a manifold.

Definition 2.1. Let M be a smooth manifold. We write $\mathrm{Diff}(M)$ to mean the group $\{f \colon M \to M : f \text{ is a diffeomorphism}\}$, implicitly with the operation $\circ : \mathrm{Diff}(M)^2 \to \mathrm{Diff}(M)$ which is composition of diffeomorphisms as functions.

It is not difficult to verify that this is a group. Associativity of composition is simply a property of functions in general, f^{-1} is a diffeomorphism if f is by definition because f is smooth and bijective with smooth inverse, and the function $\mathrm{id} : M \to M$ defined by $x \mapsto x$ is a diffeomorphism and an identity with respect to function composition. $\mathrm{Diff}(M)$ is an interesting group in its own right, but in fact it has great significance if one wishes to study the manifold M itself, given the following theorem of Filipkiewicz [7]:

Theorem 2.2 (Filipkiewicz)**.** *Let M_1, M_2 be connected smooth manifolds. If there exists an isomorphism $\varphi : \mathrm{Diff}(M_1) \to \mathrm{Diff}(M_2)$, then there exists a diffeomorphism $w : M_1 \to M_2$ such that $\varphi(f) = wfw^{-1}$ for all $f \in \mathrm{Diff}(M_1)$.*

In other words, we have a version of Felix Klein's Erlangen program for smooth manifolds, in that the algebraic structure of the symmetries of the manifold entirely determines the geometric structure of the manifold itself. Understanding the structure, then, of $\mathrm{Diff}(M)$ seems an important task. As it turns out, $\mathrm{Diff}(M)$ has a manifold-esque structure itself, but we will not yet make this precise.

In general, an informative topology associated to a group (or other algebraic structure) can be very helpful in studying that group. A standard motivating example for this is the group of reals $\mathbb{R}$ under addition, where the topology is the standard one on $\mathbb{R}$. For instance, if we take any group-theoretic reasoning or property for which generating sets are relevant, it is useful to note that any open neighborhood of the identity generates the entire group (because of the Archimedean property). The reason that the topology on $\mathbb{R}$ is useful in studying the algebraic group is that the topology and group operation are compatible in that addition and taking inverses are continuous operations. More generally, we may define a *topological group* to be a group G and a topology $\mathcal{T}$ on G such that for every $x \in G$, the maps $y \mapsto y^{-1}$, $y \mapsto xy$, and $y \mapsto yx$ are continuous with respect to $\mathcal{T}$. Given this, we may in fact generalize our earlier Archimedean property statement in the following way:

Lemma 2.3. *Let $U \subseteq G$ be an open subset of a connected topological group G such that $e \in U$. Then U generates G.*

Proof. We show that $\langle V \rangle$, the subgroup generated by $V = U \cap U^{-1} \subseteq U$, is nonempty, open, and closed. From those three properties, connectedness of G forces $\langle V \rangle$ to be the entirety of G. By hypothesis, e is in U and thus in V, so V is nonempty. Note that $y \mapsto gy$ is bijective and bicontinuous for any $g \in G$, so it is a homeomorphism. Thus, gV is open for any $g \in \langle V \rangle$, so $\langle V \rangle$ is open because it consists of words in V. We move on to closedness to finish the proof. If $g \notin \langle V \rangle$, then gV is open, contains g, and is disjoint from $\langle V \rangle$, so $G \setminus \langle V \rangle$ is open. $\square$

Because multiplication by an element acts as a homeomorphism on the group, any neighborhood U of the identity can be taken to a homeomorphic neighborhood around any element x via multiplication by x. Therefore, topological groups must be locally uniform in this sense. Given this local uniformity, a natural class of topological groups to ask about is those which are topological manifolds. Consider the set $\mathrm{Mat}(n, \mathbb{R})$, the set of $n \times n$ matrices over $\mathbb{R}$ under addition, and take the subset $\mathrm{GL}(n, \mathbb{R})$ (for "General Linear") of invertible elements of $\mathrm{Mat}(n, \mathbb{R})$ under matrix multiplication. This subset will be a group (because matrix multiplication is associative, I is the identity, and inverses exist by construction). We can treat this as an abstract group, i.e., a set with a binary function, and abstractly it is an important group; however, if we consider each $A \in \mathrm{GL}(n, \mathbb{R})$ as an element of Euclidean space $\mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$, the topology inherited from $\mathbb{R}^{n^2}$ makes $\mathrm{GL}(n, \mathbb{R})$ a topological group homeomorphic to a manifold (the fact that the operations are continuous is not difficult, but it is nontrivial that it the group is a manifold). Given that we now have manifolds in play, it is natural to ask whether or not we may talk about groups which are smooth manifolds.

A *Lie group* is a group G together with a compatible smooth manifold structure $(U_i, \varphi_i)_{i \in I}$ such that the aforementioned maps are smooth.

Motivating examples of topological, but especially Lie, groups often come from groups of matrices such as $\mathrm{GL}(n,\mathbb{R})$, though again it is nontrivial that $\mathrm{GL}(n,\mathbb{R})$ may be given a smooth structure at all. As we will come back to later, the subgroup $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det(A) = 1\}$, and its quotient $\mathrm{PSL}(n,\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/Z(\mathrm{SL}(n,\mathbb{R}))$ are both Lie groups by taking the inherited (from $\mathrm{GL}(n,\mathbb{R})$) and quotient smooth structure respectively. Here, $Z(\mathrm{SL}(n,\mathbb{R}))$ denotes the center, and is equal to either $\{I\}$ or $\{I,-I\}$ for odd and even n respectively.

Unfortunately, $\mathrm{Diff}(M)$ is far too large for a manifold structure. The natural topology on it is the compact-open topology, which has as a basis sets of the form $\mathcal{N}(U_i, U_{i'}, K, \epsilon, f)$ with U_i and $U_{i'}$ sets in the atlas of M with charts (U_i, φ_i) and $(U_{i'}, \varphi_{i'})$, K a compact subset of M, $\epsilon > 0$, and $f \in \mathrm{Diff}(M)$, where $g \in \mathcal{N}(U_i, U_{i'}, K, \epsilon, f)$ if and only if $g(K) \subseteq V$ and $\|D^k[\varphi_{i'} g \varphi_i^{-1}](x) - D^k[\varphi_{i'} f \varphi_i^{-1}](x)\| \lneq \varepsilon$ for all $x \in K$ and $k \in \mathbb{N}$. The topology this basis induces turns out to be an infinite-dimensional manifold of sorts: locally, it is homeomorphic to a Fréchet space, though the details of precisely what Fréchet spaces or infinite-dimensional manifolds are largely unimportant to us. It will suffice to consider the neighborhood of a diffeomorphism to be an infinite-dimensional, separable, complete normed vector space. Still, this topology on $\mathrm{Diff}(M)$ is larger than what we would like. In fact, another failing of this topology is that the behavior at infinity, so to speak, cannot be accounted for, because individual neighborhoods only take into account compact sets.

However, if we define the support of $g \in \mathrm{Diff}(M)$, denoted $\mathrm{supp}(g)$, to be the closure of the set $\{x \in M : g(x) \neq x\}$, the set $\{g \in \mathrm{Diff}(M) : \mathrm{supp}(g) \text{ is compact}\}$ is a normal subgroup. This is not only significantly smaller, but also has a nicer metric (see Equation 2.1) which we will consider further later, and avoids the behavior at infinity issue. Recalling that the (path) component of $x \in X$ for a topological space X is the maximum (path-)connected subset of X containing x with respect to inclusion, we may consider the (path) component of $\mathrm{id} \in \mathrm{Diff}(M)$. The path component and component are, in the case of many groups (including Lie groups and $\mathrm{Diff}(M)$) the same set, and we will denote the component of the identity in $\mathrm{Diff}(M)$ by $\mathrm{Diff}_0(M)$. The subgroup $\mathrm{Diff}_0(M)$ is normal, and its structure is very closely related to $\mathrm{Diff}(M)$, so to prove structural theorems about $\mathrm{Diff}(M)$ one often considers $\mathrm{Diff}_0(M)$. In particular, one reason to consider $\mathrm{Diff}_0(M)$ and its subgroups is Lemma 2.3. $\mathrm{Diff}_0(M)$, while nicer in several ways than the full $\mathrm{Diff}(M)$, still suffers from many issues topologically as well as the issue of behavior at infinity, so we will actually mostly consider the subgroup $\mathrm{Diff}_c(M) = \{g \in \mathrm{Diff}_0(M) : \mathrm{supp}(g) \text{ is compact}\}$. Of course, when M is compact, $\mathrm{Diff}_c(M) = \mathrm{Diff}_0(M)$. In general, the relationship between $\mathrm{Diff}(M)$ and $\mathrm{Diff}_0(M)$ is paralleled by that of $\mathrm{Diff}_0(M)$ and $\mathrm{Diff}_c(M)$

in that proving theorems about the latter often gives us theorems about the former. Two results of Thurston [19] exemplify this,[1] where we recall that a group is called perfect if it equals its own commutator subgroup:

Lemma 2.4 (First Thurston Lemma)**.** *If* $\mathrm{Diff}_c(M)$ *is perfect as a group, then* $\mathrm{Diff}_0(M)$ *is simple.*

Lemma 2.5 (Second Thurston Lemma)**.** *For* $U \subseteq M$ *with the closure of* U *compact, let* G_U *denote the normal subgroup of* $\mathrm{Diff}_c(M)$ *of elements* g *with* $\mathrm{supp}(g)$ *contained in* U.[2] *Then* $G_U \subseteq \mathrm{Diff}_c(M)$ *is perfect if and only if* $\mathrm{Diff}_0(M)$ *is perfect.*

Actually, simplicity is quite a strong algebraic condition on a group, so by Theorem 2.2, perfectness of $\mathrm{Diff}_c(M)$ gives us useful information about M. In fact, the simplicity of $\mathrm{Diff}_0(M)$ was a conjecture of Smale [18] resolved by Thurston using these lemmas. However, it was was an essential input to Thurston's proof of simplicity that $\mathrm{Diff}_c(M)$ was perfect, and a more accessible one by far than the proofs of Thurston's lemmas. This motivates what will become our central theorem:

Theorem 2.6. *For every smooth manifold* M *of dimension at least 2,* $\mathrm{Diff}_c(M)$ *is perfect.*

There are a number of important things to note here. First, the theorem does hold true for M of dimension 1, though having dimension at least 2 is necessary for the proof we give here. Second, we do prove the theorem for $M = S^1$, the circle. Finally, proving the theorem for S^1 is, in a rigorous sense, as close as we can get to proving the theorem for dimension 1 without fully proving it. This is because of the well-known fact that every 1-dimensional smooth manifold is diffeomorphic to a disjoint union of copies of S^1 and $\mathbb{R}$, as well as the additional fact that diffeomorphisms in the same connected component of $\mathrm{Diff}(M)$ as id must preserve connected components of M. From these, we can conclude that $\mathrm{Diff}_c(M)$ with M 1-dimensional is isomorphic to the direct product $(\mathrm{Diff}_c(S^1))^n \times (\mathrm{Diff}_c(\mathbb{R}))^m$ for some $n, m \in \mathbb{N}$.

Earlier, we referred to path-connectedness in $\mathrm{Diff}(M)$. Given that we only currently have a topology on $\mathrm{Diff}(M)$, defining smoothness requires a different perspective on a path from the usual continuous function $[0, 1]$ to $\mathrm{Diff}(M)$. We give this alternative perspective.

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[1] Together with a lemma we will prove in the next section, Lemma 3.2, and several lemmas not relevant to our exposition, these comprise what Banyaga calls the "Thurston tricks" [2].

[2] We note that for every G_U, as well as every G_K where $G_K = \{g \in \mathrm{Diff}_c(M) : \mathrm{supp}(g) \subseteq K\}$ for compact K, Lemma 2.3 applies.

Definition 2.7. An isotopy is a smooth function $\phi : M \times [0,1] \to M$ such that for every $t \in [0,1]$, $\phi_t(x) : x \mapsto \phi(x,t)$ is a diffeomorphism of M.

This intuitively aligns with what a smooth path in $\mathrm{Diff}(M)$ should be. We claim something stronger, though: that isotopies characterize what smooth functions on $\mathrm{Diff}(M)$ are entirely. Traditionally, we would define smoothness via the infinite-dimensional manifold structure that we mentioned in passing earlier, but for convenience, we give a characterization which is well-known to be equivalent.

Definition 2.8. A function $F : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is smooth if and only if, for every isotopy $(\phi_t)_{t\in[0,1]}$, $F(\phi_t)$ is an isotopy.

Our last piece of preliminary information is the metric on $\mathrm{Diff}_c(M)$ that we mentioned earlier. This is, in full generality, a metric on $C_c^\infty(M,M)$, the C^∞ functions from M to M with compact support. It is defined as follows:

$$\delta(f,g) = \sup_{\substack{x\in M\\ k\in\mathbb{N}}} \sup_{x\in U_i, U_{i'}} \|D^k[\varphi_{i'} g \varphi_i^{-1}](x) - D^k[\varphi_{i'} f \varphi_i^{-1}](x)\|. \tag{2.1}$$

Balls are of course defined accordingly. The following, which we will invoke later, is more or less a consequence of the Inverse Function Theorem.

Lemma 2.9. *$\mathrm{Diff}_c(M)$ is open in $C_c^\infty(M,M)$.*

3. Reduction to Single Manifolds

Our overall goal for this paper will be to prove Theorem 2.6. Unless stated otherwise, everything in the sequel is an adaptation of [14] with numerous interludes for differential-geometric background (whose proofs are largely adapted from [2, 13]). The bulk of our efforts in this paper will be devoted to the case $M = \mathbb{R}^n$ for $n \geq 2$. In this section, we aim to prove that this case is enough.

Proposition 3.1. *For any n-dimensional smooth manifold M with $n \geq 2$, the group $\mathrm{Diff}_c(M)$ is perfect if $\mathrm{Diff}_c(\mathbb{R}^n)$ is.*

In turn the bulk of this reduction will consist of proving the Fragmentation Lemma, which is an important tool when discussing the structure of diffeomorphism groups. Our exposition of the proof is one which follows the structure of that of [2], but we have adapted it to be accessible to a reader with far less background.

Lemma 3.2 (Fragmentation Lemma)**.** *Let M be a smooth manifold, $(W_l)_{l\in L}$ an open cover of M for some index set L, and g any element of $\mathrm{Diff}_c(M)$. Then g can be written as a product $g_1 \circ g_2 \circ \cdots \circ g_n$ of elements of $\mathrm{Diff}_0(M)$ such that for every $1 \leq j \leq n$, $\mathrm{supp}(g_j) \subseteq W_l$ for some $l \in L$.*

However, before we can move onto the proof of Lemma 3.2, we must introduce a tool. We say that a collection of open subsets $(U_i)_{i\in I}$ of a smooth manifold M is locally finite if for every $x \in M$, the set $\{i \in I : x \in U_i\}$ is finite.

Definition 3.3. Let $(U_i)_{i\in I}$ be a locally finite open cover of a smooth manifold M for some index set I. We define a partition of unity subordinate to $(U_i)_{i\in I}$ to be a collection of functions $(\psi_i)_{i\in I}$ such that:

(1) ψ_i is a smooth function M to $\mathbb{R}$ for every $i \in I$,
(2) $0 \leq \psi_i(x) \leq 1$ for every $i \in I$ and $x \in M$,
(3) $\text{supp}(\psi_i) \subseteq U_i$ for every $i \in I$, and
(4) $\sum_{i\in I} \psi_i(x) = 1$ for every $x \in M$.

This notion is useful very broadly for differential geometry. The idea behind it is that we may take locally defined notions (such as an integral over a part of a manifold) and build them up from local pieces to make them global (which continuing the integral example would be defining integrals for functions whose domains are the entire manifold). For us, we will use them to gradually build up a diffeomorphism whose support is contained in some entire compact set from diffeomorphisms that only affect smaller parts of that compact set. The following theorem, which we will prove later, tells us that for any atlas we can define a partition of unity. One should keep in mind that, by the local compactness of manifolds, any atlas may be reduced to a locally finite one.

Theorem 3.4 (Existence of Partitions of Unity)**.** *Let $(U_i)_{i\in I}$ be a locally finite open cover of a manifold M with index set I. Then there exists a partition of unity subordinate to $(U_i)_{i\in I}$.*

With this preparation out of the way, we move on to the proof of the Fragmentation Lemma:

Proof of Lemma 3.2. Recall our smooth manifold M, our diffeomorphism g in $\text{Diff}_c(M)$, and our open cover $(W_l)_{l\in L}$ with index set L. We will instead consider for most of this proof the collection $(U_i)_{i\in I}$, indexed by some I, of open sets which are contained in some W_l and whose closures are compact. If we can prove the theorem for $(U_i)_{i\in I}$, it follows for $(W_l)_{l\in L}$, because having support contained in some U_i implies having support contained in some W_l.

First, note that $\text{supp}(g)$ is compact, so there is a finite $I' \subseteq I$ such that $\text{supp}(g) \subseteq \bigcup_{i\in I'} U_i$. Letting $U = \bigcup_{i\in I'} U_i$, consider some isotopy g_t from $g_0 = \text{id}$ to $g_1 = g$ contained in $G_U \subseteq \text{Diff}_c(M)$. For any r, we can write g as

$$g = (g_0^{-1} g_{1/r}) \circ (g_{1/r}^{-1} g_{2/r}) \circ \cdots \circ (g_{r-1/r}^{-1} g_1).$$

Notice that for such an r, we only need to provide the fragmentation for each $g_{r,k} = g^{-1}_{(k-1)/r} g_{k/r}$ factor. We claim that for any open neighborhood $\mathcal{V}$ of id in $\mathrm{Diff}_c(M)$, there exists an $r_{\mathcal{V}}$ such that $g_{r,k}$ is in $\mathcal{V}$ for every $1 \leq k \leq r$, or in other words, that we may make the factors we are considering as close to id as we like. To prove this, recall that the isotopy g_t is a path γ in G_U, so we may consider (by a slight abuse of notation) its image $\gamma = \{g_t : t \in [0,1]\}$. Define the family of sets $\mathcal{V}_{g_t} = \{f \in \mathrm{Diff}_c(M) : f g_t^{-1}, g_t f^{-1} \in U \cap U^{-1}\}$. Then $\mathcal{V}_{g_t}$ is open in $\mathrm{Diff}_c(M)$ for every t, so $\mathcal{V}_{g_t} \cap \gamma$ is by definition open in γ. The path γ is a homeomorphism onto its image, so $\gamma^{-1}(\mathcal{V}_{g_t})$ is open in $[0,1]$, and there must be some $q \in \gamma^{-1}(\mathcal{V}_{g_t})$ such that $q \in \mathbb{Q}$. Therefore, $g_q \in \mathcal{V}_{g_t}$, and because the relation defining $\mathcal{V}_{g_t}$ is symmetric, we have $g_t \in \mathcal{V}_{g_q}$. Thus, $(\mathcal{V}_{g_q})_{q \in \mathbb{Q}}$ is an open cover of γ. By compactness, there must be some finite $T \subseteq \mathbb{Q}$ such that $(\mathcal{V}_{g_q})_{q \in T}$ also covers γ. There must be some r_T such that every element of T is expressible with denominator r_T, so we take $r_{\mathcal{V}} = r_T$, yielding the claim.

With this in hand, we may proceed to prove the lemma for each $g_{r,k}$ term, using the fact that $g_{r,k}$ is arbitrarily close to the identity. Informally, doing this will consist of building up $g_{r,k}$ successively on each of the elements of the finite cover $(U_i)_{i \in I'}$. We use partitions of unity to make this gradation idea rigorous. By Theorem 3.4, if U is considered as a smooth manifold itself, then there is a partition of unity $(\psi_i)_{i \in I'}$ subordinate to the open cover $(U_i)_{i \in I'}$ of U. Denoting $J = \{1, 2, \ldots, |I'|\}$, we re-index $(U_i)_{i \in I'}$ as $(U_j)_{j \in J}$, allowing us to define

$$\mu_m = \sum_{\substack{j \in J \\ j \leq m}} \psi_j.$$

Take some isotopy $(g_{r,k})_t$ in $\mathrm{Diff}_c(M)$ from $(g_{r,k})_0 = \mathrm{id}$ to $(g_{r,k})_1 = g_{r,k}$, which we may construct such that for every t, $(g_{r,k})_t$ is arbitrarily close to id in the same sense that $g_{r,k}$ is. Then we may define, for $m \in J$, $f_{m,r,k}(x) = (g_{r,k})_{\mu_m(x)}(x)$. The idea here is roughly what we described when we introduced partitions of unity: to get from $f_{m-1,r,k}$ to $f_{m,r,k}$, we partially apply $g_{r,k}$ on U_m, and so in each iteration we build a function closer to $g_{r,k}$. In symbols,

$$f_{m,r,k}(x) = [(g_{r,k})_{\psi_m(x)} \circ f_{m-1,r,k}](x).$$

As we mentioned earlier, for any neighborhood $\mathcal{V}$ of id in $\mathrm{Diff}_c(M)$, we can pick an $r \in \mathbb{N}$ and an isotopy $(g_{r,k})_t$ such that each function in the isotopy $(g_{r,k})_t$ is in $\mathcal{V}$, so each $f_{m,r,k}$ is as well, because $f_{m,r,k}$ is closer in $C^\infty_c(M,M)$ to id than $(g_{r,k})_{\max_x \mu(x)}$. Thus, because the set $C = \{f \in C^\infty_c(M,M) : f \text{ has a smooth inverse}\}$ is open in $C^\infty_c(M,M)$ by Lemma 2.9, we may choose r and our isotopies such that each $f_{m,r,k}$ is contained in C. Taking this openness of C for granted for now, $f_{m,r,k}$ is in $\mathrm{Diff}_c(M)$, has support contained in $\text{supp}(g)$ and thus compact support, and $\gamma_t = (g_{r,k})_{t\cdot\psi_m(x)}(x)$ is an isotopy from $f_{m,r,k}$ to id, so $f_{m,r,k}$ is in $\text{Diff}_c(M)$. Therefore, we may write

$$\begin{aligned} g_{r,k}(x) &= f_{|I'|,r,k}(x) \\ &= [(g_{r,k})_{\psi_{|I'|}(x)} \circ \cdots \circ (g_{r,k})_{\psi_1(x)}](x) \\ &= [(f_{|I'|,r,k} f^{-1}_{|I'|-1,r,k}) \circ \cdots \circ (f_{m,r,k} f^{-1}_{m-1,r,k}) \circ \cdots \circ (f_{1,r,k}\,\text{id})](x) \end{aligned}$$

Because $(g_{r,k})_{\psi_m(x)}(x) = x$ when $x \notin U_m$, $\text{supp}(f_{m,r,k} f^{-1}_{m-1,r,k}) \subseteq U_m$, and $f_{m,r,k} f^{-1}_{m-1,r,k}$ is an element of $\text{Diff}_c(M)$, so this gives us a fragmentation of $g_{r,k}$. The fragmentation of g follows from the fragmentation of each $g_{r,k}$, as noted previously, so we are done. □

For the existence of partitions of unity, we use a similar idea to the proof in [13], though that proof appears somewhat far past the background we are assuming from the reader.

Proof of Theorem 3.4. Recall our manifold M and our locally finite open cover $(U_i)_{i\in I}$.

A smooth function $h : \mathbb{R}^n \to \mathbb{R}$ is a bump function if there exist positive reals $a \gneq b \gneq 0$ such that

(1) if $\|x\| \geq a$, then $h(x) = 0$,
(2) if $a \gneq \|x\| \gneq b$, then $0 \lneq h(x) \lneq 1$, and
(3) if $\|x\| \leq b$, then $h(x) = 1$.

We will demonstrate such an h for every n, a, b later, but for now take it to exist. We have our smooth n-dimensional manifold M with atlas $(V_j, \varphi_j)_{j\in J}$ indexed by some set J and our locally finite open cover $(U_i)_{i\in I}$. Construct the open cover

$$\{W \subseteq M : W \subseteq U_i \cap V_j \text{ for some } i \in I, j \in J \\ \text{and } \varphi(W) = B_r(x) \text{ for some } r \in \mathbb{R}, x \in \mathbb{R}^n\},$$

and take a locally finite subcover $(W_l)_{l\in L}$ indexed by L. For each $l \in L$ we can find $j \in J$, $x_l \in \mathbb{R}$, and a bump function h_l such that $\varphi_j(W) + x$ is the support of h_l. We then define γ_l as $h \circ (\varphi_j + x_l)$ on V_j and 0 elsewhere. From this we may define

$$\psi_i'(x) = \sum_{\substack{x\in W_l \\ W_l \subseteq U_i}} \gamma_l(x),$$

and our claimed partition of unity is

$$\psi_i(x) = \frac{\psi_i'(x)}{\sum_{x\in U_i} \psi_i'(x)}.$$

Each of our functions is well-defined, because the open covers are locally finite and thus the sums are finite for each x. Smoothness of ψ_i, $\operatorname{supp}(\psi_i) \subseteq U_i$, and $\sum_{i\in I} \psi_i(x) = 1$ are obvious. It is also clear that each ψ_i is nonnegative, so we also have $0 \leq \psi_i(x) \leq 1$.

All that is left is our bump functions. We can define the auxiliary function

$$h(x) = \begin{cases} \exp(-1/x) & x \geq 0 \\ 0 & x \lneq 0, \end{cases}$$

and from this we can define a bump function for arbitrary n, a, b:

$$g_{a,b}(x) = \frac{h(a - \|x\|)}{h(a - \|x\|) + h(\|x\| - b)}.$$

It is not difficult to verify that this satisfies the conditions of a bump function, so we are done. □

Given the Fragmentation Lemma, our main result of the section follows fairly naturally:

Proof of Proposition 3.1. At a high level, we construct an isomorphism between G_U and $\operatorname{Diff}_c(\mathbb{R}^n)$ for every open U diffeomorphic to a ball, and then use the Lemma 3.2 to reduce perfection of $\operatorname{Diff}_c(M)$ to perfection of each G_U. In more detail, suppose that $\operatorname{Diff}_c(\mathbb{R}^n)$ is perfect, that $(U_i, \varphi_i)_{i\in I}$ is a chart of M, and that $(V_j)_{j\in J}$ is an indexing of the set $\{V : V \subseteq U_i \text{ and } \varphi_i(V) = B_r(x) \text{ for some } i, r, x\}$. Then we take a locally finite subcover $(V_j)_{j\in J'}$ of M indexed by $J' \subseteq J$. For each V_j there is a diffeomorphism $\gamma_j : \varphi(V_j) \to \mathbb{R}^n$. V_j is a smooth manifold in its own right, and $f \mapsto f|_{V_j}$ is an isomorphism from G_{V_j} to $\operatorname{Diff}_c(V_j)$. In turn, the map $f \mapsto (\varphi_i \circ \gamma_j) f (\varphi_i \circ \gamma_j)^{-1}$ is an isomorphism from $\operatorname{Diff}_c(V_j)$ to $\operatorname{Diff}_c(\mathbb{R}^n)$. Thus, because $\operatorname{Diff}_c(\mathbb{R}^n)$ is perfect, G_{V_j} is for each V_j, so $f \in G_{V_j}$ is a product of commutators $[f_1, f_2] \ldots [f_{n-1}, f_n]$. By Lemma 3.2, any $g \in \operatorname{Diff}_c(M)$ is a product $g = g_1 \circ \cdots \circ g_n$ with $g_k \in G_{V_{j_k}}$ for every k and some j_k. Thus, we conclude that

$$\begin{aligned} g &= g_1 \circ \cdots \circ g_n \\ &= ([g_{1,1}, g_{1,2}] \ldots [g_{1,m_1-1}, g_{1,m_1}]) \circ \cdots \circ ([g_{n,1}, g_{n,2}] \ldots [g_{n,m_n-1}, g_{n,m_1}]), \end{aligned}$$

and $\operatorname{Diff}_c(M)$ is perfect. □

4. Diffeomorphisms of the Circle

The main result of this section will be, aside from some technical details, that $\operatorname{Diff}_c(S^1)$ is perfect. It may not be immediately obvious to the reader why S^1 is relevant at all, but in fact it will be essential to our argument for the perfectness of $\mathrm{Diff}_c(\mathbb{R}^n)$. To even properly state our result, though, will require some background.

4.1. **Vector Fields,** $\mathrm{PSL}(2,\mathbb{R})$**, and the Theorem Statement.** We begin by giving an introduction to exponentials of vector fields. Recall that the tangent bundle TM of a manifold M is the disjoint union $\bigsqcup_{x\in M} T_xM = \{(x,v) : x \in M, v \in T_xM\}$ together with the linear structure of T_xM on each fiber $\{x\} \times T_xM$.

Definition 4.1. Let M be a smooth manifold. A smooth vector field on M is a smooth function $X : M \to TM$ such that $X(x)$ is in $\{x\} \times T_xM$, and the support of X is the closure of the set $\{x \in M : f(x) \neq 0\}$. The set of smooth vector fields on M is denoted $\mathfrak{X}(M)$, and the set of smooth vector fields on M whose support is compact is denoted $\mathfrak{X}_c(M)$.

We will, for the most part, identify an element $(x,v) \in \{x\}\times T_xM$ simply with v.

Now, this definition of support more closely aligns with what the reader is likely familiar with from previous experiences, such as real or complex analysis, and it is not immediately apparent that this is particularly related to our notion of the support of a diffeomorphism. However, as we will show, the support of a vector field and the support of a diffeomorphism are closely related. This relationship is due to the fact that a smooth vector field induces a smooth diffeomorphism in the following manner.

We say that a function $\phi : M \times \mathbb{R} \to M$ is a *flow* if $\phi(x,0) = x$ and $\phi(\phi(x,s),t) = \phi(x,s+t)$ for $x \in M$ and $s,t \in \mathbb{R}$. If we have a function $\phi : M \times \mathbb{R} \to M$ which is smooth and a flow, then intuitively it is a way to move around M for some time t such that moving according to ϕ for time s and then time t is the same as simply moving according to ϕ for time $s+t$. For any smooth vector field $X \in \mathfrak{X}_c(M)$, we can define a flow $\phi_X(x,t)$ via the (autonomous) differential equation

$$\frac{d}{dt}\phi_X(x,t) = X(\phi_X(x,t))$$

with initial condition $\phi_X(x,0) = x$. While the theorem of existence and uniqueness tells us that a solution exists locally, it is nontrivial that a solution to this differential equation exists globally. We give the broad intuition for why this is true, as a proper proof is a little long and the existence is intuitively obvious. We can deduce from the aforementioned existence and uniqueness theorem that for every $x \in M$ there must exist a unique solution on some open $U_x \subseteq M \times \mathbb{R}$ containing $(x,0)$, so there must exist an open $U \subseteq M$ containing x and an $\varepsilon_x > 0$ such that there exists a unique solution on $U'_x \times (-\varepsilon_x, \varepsilon_x)$. By compactness of the support of X, there must exist a finite set $I \subseteq M$ such that $\bigcup_{x\in I} U'_x$ contains the support of M. Therefore, if we denote $\varepsilon_M = \min(\{\varepsilon_x : x \in I\})$, there is a unique solution for every x with $t \in (-\varepsilon_M, \varepsilon_M)$. Finally, we may extend this uniform time solution to a global solution by, for each $x \in M$, taking the curve

$$\gamma_x(t) = \begin{cases} \phi_X(x,t) & t \in (-\varepsilon_M, \varepsilon_M) \\ \phi_X(\phi_X(x,\varepsilon_M), t-\varepsilon_M) & t \in (0,\infty) \\ \phi_X(\phi_X(x,-\varepsilon_M), t+\varepsilon_M) & t \in (-\infty, 0). \end{cases}$$

This last step in particular requires some lengthy justification, but these details are not so important to us, and for the curious reader, Theorem 9.16 of [13] makes much more precise and detailed all of the above.

We may denote the time t map of the flow $\phi_X(x,t)$, for fixed t, by $\phi_X^{(t)}$, so the time one map of the flow is $\phi_X^{(1)}$. We call the function $\exp : \mathfrak{X}_c(M) \to \mathrm{Diff}_c(M)$ $X \mapsto \phi_X^{(1)}$ the exponential map. In fact it takes a little bit of thought not just to prove well-definedness of exp, but also to prove that the image of exp is $\mathrm{Diff}_c(M)$. Our first consideration is support. We claimed earlier that the support of the vector field is related to the support of the corresponding diffeomorphism, and in fact the support of X is equal to the support of $\exp(X)$, because when $0 = X(x) = X(\phi_X(y,1))$ for some x, y, then $\frac{d}{dt}\phi_X(x,t)\Big|_{t=1} = 0$ and $\phi_X^{(1)}$ is constant at x. For the isotopy to the identity, simply consider $\gamma_t = \exp(tX)$.

(Some readers with more background might notice the similarities to the exponential for a Riemannian manifold or, much more importantly to the study of diffeomorphism groups, the analogue of the exponential for finite-dimensional Lie groups. In more detail about that remark, $\mathfrak{X}_c(M)$ is the Lie algebra of $\mathrm{Diff}_c(M)$, and in spite of the fact that the exponential for $\mathrm{Diff}_c(M)$ is neither injective nor surjective onto any neighborhood of the identity, the nice properties of the elements of the image of the exponential make these elements and the exponential important to the study of $\mathrm{Diff}_c(M)$ generally and to our proof in particular.)

We may now, at least, state our main theorem for the section in full.

Theorem 4.2. *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(S^1) = \mathrm{Diff}_c(S^1)$ such that any $g \in \mathcal{U}$ can be written as a product of four commutators $g = [G_1(g), f_1] \dots [G_4(g), f_4]$, with f_i independent of g and each $G_i : \mathrm{Diff}_0(S^1) \to \mathrm{Diff}_0(S^1)$ smooth. Further, we may take $G_i(\mathrm{id}) = \mathrm{id}$ and $f_i = \exp(F_i)$ with each F_i in $\mathfrak{X}_c(S^1)$.*

Before beginning the proof of Theorem 4.2, we introduce the notable subgroup $\mathrm{PSL}(2,\mathbb{R}) \subsetneq \mathrm{Diff}_0(S^1)$, which is isomorphic as a group to the quotient group $\mathrm{SL}(2,\mathbb{R})/\{I, -I\}$. However, in this inclusion, the action of $\mathrm{PSL}(2,\mathbb{R})$ on S^1 is not the action of matrices on $\mathbb{R}^2$, but instead the action of *Möbius transformations*. Here, Möbius transformations are functions $z \mapsto \frac{az+b}{cz+d}$, where z is an element of $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$. The action is defined by the aforementioned fractional linear transformation when $cz + d \neq 0$ and $z \neq \infty$; when $cz + d = 0$, $z = \infty$ and $c \neq 0$, or $z = \infty$ and $c = 0$, we have $z \mapsto \infty$, $z \mapsto \frac{a}{c}$, and $z \mapsto \infty$ respectively. Consider the inclusion $C_0 : S^1 \to \mathbb{C}$, the *Cayley transform* $C_1 : \mathbb{RP}^1 \to \mathbb{C}$ defined by $z \mapsto \frac{z-i}{z+i}$ when $z \neq \infty$ and $z \mapsto 1$ when $z = \infty$, and the identification of A with its Möbius transformation $M(A)$. The action of $[A] \in \mathrm{PSL}(2,\mathbb{R})$ on $x \in S^1$ is defined by

$$[A](x) = [C_0^{-1} \circ C_1 \circ M(A) \circ C_1^{-1} \circ C_0](x)$$

Notice that this is well defined because A and $-A$ produce the same action this way. One reason we wish to focus on these is that this is, as mentioned earlier, a group action of $\mathrm{PSL}(2,\mathbb{R})$, which is to say that if $[A]$ is the equivalence class of A in the quotient $\mathrm{SL}(2,\mathbb{R})/\{I, -I\}$, then $[AB] = [A] \circ [B]$ and $[A^{-1}] = [A]^{-1}$. This will allow us to make explicit a number of constructions and computations. Another reason for this is that

$$[A_\theta] = \left[\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right]$$

acts on S^1 via rotation by -2θ:

$$\begin{aligned} [C_1 \circ M(A_\theta) \circ C_1^{-1}](x) &= \frac{(\cos\theta - i\sin\theta)C_1^{-1}(x) + (-\sin\theta - i\cos\theta)}{(\cos\theta + i\sin\theta)C_1^{-1}(x) + (-\sin\theta + i\cos\theta)} \\ &= \frac{\exp(-i\theta)C^{-1}(x) - i\exp(-i\theta)}{\exp(i\theta)C_1^{-1}(x) + i\exp(i\theta)} \\ &= \exp(-2i\theta)\frac{C_1^{-1}(x) - i}{C_1^{-1}(x) + i} \\ &= \exp(-2i\theta)x. \end{aligned}$$

(Strictly speaking, these manipulations only hold when $C_1^{-1}(x) \neq \infty$, but because $C_1 \circ M(A_\theta) \circ C_1^{-1}$ and thus $[A_\theta]$ are smooth, we must still have $[C_1 \circ M(A_\theta) \circ C_1^{-1}](1) = \exp(-2i\theta)$.)

It is easy to show that $\mathrm{PSL}(2,\mathbb{R})$ is contained in $\mathrm{Diff}(S^1)$. Note that $\mathrm{PSL}(2,\mathbb{R})$ is connected because $[A] \mapsto \det(A)$ is continuous with connected image, and that $\mathrm{PSL}(2,\mathbb{R})$ is locally path-connected because it is locally Euclidean. Every connected and locally path-connected space is path-connected, so every element of $\mathrm{PSL}(2,\mathbb{R})$ has an isotopy to $[I] = \mathrm{id} \in \mathrm{Diff}(S^1)$ inside of $\mathrm{PSL}(2,\mathbb{R})$, and we have the inclusion $\mathrm{PSL}(2,\mathbb{R}) \subseteq \mathrm{Diff}_c(S^1)$.

An indispensable tool in our proof of Theorem 4.2 will be a result of Herman, Corollary 5.2 of [11]. Unfortunately, the proof of Herman's theorem is too deep for this paper, so we do not cover it here, though it is proved in English as Theorem 2.3.3 of [2].

Theorem 4.3 (Herman)**.** *Denote by R_θ the rotation of S^1 by θ. There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(S^1)$ and a dense set $A \subseteq [0, 2\pi)$ such that for any $\mu \in A$, any $g \in \mathcal{U}$ can be written as $R_{\lambda(g)}[G_0(g), R_\mu]$, with $\lambda : \mathrm{Diff}_c(M) \to [0, 2\pi)$ and $G_0 : \mathrm{Diff}_c(M) \to \mathrm{Diff}_c(M)$.*

4.2. **Proving $\mathrm{Diff}_c(S^1)$ is Perfect.** Armed with this, we proceed to prove our main theorem of the section.

Proof of Theorem 4.2. Consider the neighborhood $\mathcal{U}$ of Theorem 4.3. We will show that there is a neighborhood inside $\mathcal{U}$ that satisfies the conditions of Theorem 4.2. By Theorem 4.3, there exist λ, θ, and G_0 such that every element g of U can be expressed as $R_{\lambda(g,\mu)}[G_0(g, \mu), R_\mu]$ for any $\mu \in A$. From here on, we will largely omit g when writing functions of g.

We will let $G_4 = G_0$. Now, we will explicitly construct F_4 such that $R_\mu = \exp(F_4)$ and $G_1, G_2, G_3, F_1, F_2, F_3$ such that

$$R_\lambda = [G_1, \exp(F_1)][G_2, \exp(F_2)][G_3, \exp(F_3)].$$

We begin with F_4. Notice that, because S^1 is 1-dimensional, T_xS^1 is 1-dimensional subspace of $\mathbb{R}^2$ for every x (in particular, if $x = (\cos\theta, \sin\theta)$, then T_xS^1 is the span of $(\sin\theta, -\cos\theta)$). Therefore, for every x the function $y \mapsto \|y\|$ is an isometry and an isomorphism. Because of this, we can effectively consider a smooth vector field X on S^1 to be a function from S^1 to $\mathbb{R}$. To make this slightly more rigorous, for every $X \in \mathfrak{X}_c(S^1)$, we have a $Y_X \in C^\infty(S^1, \mathbb{R})$ defined by $x \mapsto \|X(x)\|$, and this Y_X corresponds to X in various ways we will expound on later. Of course, this correspondence goes the other way, and we may define for $Y \in C^\infty(S^1, \mathbb{R})$ the smooth vector field $X_Y \in \mathfrak{X}_c(S^1)$ by $x \mapsto Y(x) \cdot (\sin\theta, -\cos\theta)$. This is in fact a correspondence; i.e., $Y_{(X_Y)} = Y$ and $X_{(Y_X)} = X$.

Define $R(x, \theta) = R_\theta(x)$. One of the ways in which X_Y and Y correspond to each other is the following:

$$\phi_X^{(t)}(x) = R\left(x, \int_0^t Y_X(\phi_X^{(s)}(x))\, ds\right).$$

In other words, traveling along the flow of X is the same as rotating according to Y_X. Therefore, if we take $Y_{4,\theta} = \theta$, then

$$\begin{aligned}\exp(X_{Y_{4,\theta}})(x) &= \phi^{(1)}_{(X_{Y_{4,\theta}})}(x)\\ &= R\left(x, \int_0^1 Y_{4,\theta}(\phi^{(s)}_{X_{Y_{4,\theta}}}(x))\, ds\right)\\ &= R\left(x, \int_0^1 \theta\, ds\right)\\ &= R_\theta(x).\end{aligned}$$

Thus, if we let $F_4(g) = X_{(Y_{4,\mu})}$, then $\exp(F_4) = R_\mu$.

We now move on to the product of three commutators. We take

$$Y_1((\cos\theta, \sin\theta)) = 1 - \cos\theta$$

and

$$Y_2((\cos\theta, \sin\theta)) = -1 - \cos\theta.$$

Setting $F_1 = F_3 = X_{Y_1}$ and $F_2 = X_{Y_2}$, we have

$$\exp(F_1) = \exp(F_3) = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]$$

and

$$\exp(F_2) = \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right].$$

Also, for a nonzero angle α, define $G_\alpha = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$.

Note that $\det(G_\alpha) = \det(\exp(F_i)) = 1$ for $i = 1, 2, 3$, so we may easily compute the commutators

$$\begin{aligned}[G_\alpha, \exp(F_1)] &=\\ [G_\alpha, \exp(F_3)] &= \left[\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\right]\\ &= \left[\begin{pmatrix} \alpha & \alpha \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right]\\ &= \left[\begin{pmatrix} 1 & \alpha^2 - 1 \\ 0 & 1 \end{pmatrix}\right].\end{aligned}$$

We may compute, again fairly easily, our other commutator:

$$\begin{aligned}[G_\beta, \exp(F_2)] &= \left[\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1}\right]\\ &= \left[\begin{pmatrix}\beta & 0\\ \beta^{-1} & \beta^{-1}\end{pmatrix}\begin{pmatrix}\beta^{-1} & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\right]\\ &= \left[\begin{pmatrix}1 & 0\\ \beta^{-2}-1 & 1\end{pmatrix}\right].\end{aligned}$$

We then compute the product of commutators

$$\begin{aligned}&[G_\alpha, \exp(F_1)][G_\beta, \exp(F_2)][G_\alpha, \exp(F_3)]\\ &= \left[\begin{pmatrix}1 & \alpha^2-1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \beta^{-2}-1 & 1\end{pmatrix}\begin{pmatrix}1 & \alpha^2-1\\ 0 & 1\end{pmatrix}\right]\\ &= \left[\begin{pmatrix}1+(\alpha^2-1)(\beta^{-2}-1) & 2(\alpha^2-1)+(\alpha^2-1)^2(\beta^{-2}-1)\\ \beta^{-2}-1 & 1+(\alpha^2-1)(\beta^{-2}-1)\end{pmatrix}\right].\end{aligned}$$

Letting $\beta(\lambda) = (\sin(-\lambda(g)/2)+1)^{-\frac{1}{2}}$, if we may choose $\alpha(g)$ such that

$$-(\beta(\lambda)^{-2}-1) = 2(\alpha(\lambda)^2-1) + (\alpha(\lambda)^2-1)^2(\beta(\lambda)^{-2}-1),$$

then we have $-\lambda(g)/2 = \sin^{-1}(\beta(\lambda)^{-2}-1)$ and

$$\begin{aligned}&[G_{\alpha(\lambda)}, \exp(F_1)][G_{\beta(\lambda)}, \exp(F_2)][G_{\alpha(\lambda)}, \exp(F_3)]\\ &= \left[\begin{pmatrix}\cos(-\lambda(g)/2) & \sin(-\lambda(g)/2)\\ -\sin(-\lambda(g)/2) & \cos(-\lambda(g)/2)\end{pmatrix}\right]\\ &= R_{\lambda(g)}.\end{aligned}$$

Thus, we only need show that $\alpha(\lambda)$ can be chosen in this manner. In $(0, \sqrt{2})$, there exists such a β for each α:

$$\beta = \left(\frac{-2(\alpha^2-1)}{1+(\alpha^2-1)^2}+1\right)^{-\frac{1}{2}}.$$

By the Inverse Function Theorem, $\alpha \mapsto \left(\frac{-2(\alpha^2-1)}{1+(\alpha^2-1)^2}+1\right)^{-\frac{1}{2}} = \beta(\alpha)$ is a local diffeomorphism on some open U containing $\alpha = 1$, so we take the inverse $\alpha_0(\beta)$. The function $\beta(\lambda)$ is a local diffeomorphism on an open V containing $\lambda = 0$, and $\beta(0) = 1$, so because $\alpha_0(1) = 1$, then $\alpha(\lambda) = \alpha_0(\beta(\lambda))$ is a local diffeomorphism on $V \cap \beta^{-1}(U)$ containing $\lambda = 0$. By smoothness of λ, $\lambda^{-1}(V \cap \beta^{-1}(U))$ is open, so $\mathcal{U} \cap \lambda^{-1}(V \cap \beta^{-1}(U))$ satisfies the theorem statement. □

5. The Proof for $\mathbb{R}^n$

In this last section, we apply the perfectness of S^1 to induct on n and prove that $\mathrm{Diff}_c(\mathbb{R}^n)$ is perfect for $n \geq 2$.

Before that, though, we introduce a lemma that is more apparently applicable to our stated goal. The lemma breaks an element of $\mathrm{Diff}_c(\mathbb{R}^n)$ into pieces closer to elements of $\mathrm{Diff}_c(\mathbb{R}^{n-1})$ and $\mathrm{Diff}_c(\mathbb{R})$. If g and h are in $\mathrm{Diff}_c(\mathbb{R}^n)$, we say that g *preserves horizontal hyperplanes* and h *preserves vertical lines* if $g(\mathbb{R}^{n-1} \times \{x\}) = \mathbb{R}^{n-1} \times \{x\}$ for every x, and $h(\{v\} \times \mathbb{R}) = \{v\} \times \mathbb{R}$ for every $v \in \mathbb{R}^{n-1}$. The lemma is below.

Lemma 5.1. *There is a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_c(\mathbb{R}^n)$ such that any $f \in \mathcal{U}$ can be written as $G(f) \circ H(f)$, where $H(f)$ preserves each vertical line, $G(f)$ preserves each horizontal hyperplane, and both $G, H : \mathcal{U} \to \mathrm{Diff}_c(\mathbb{R}^n)$ are smooth.*

Proof. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denote projection to the ith coordinate; i.e., $(x_1, \ldots, x_n) \mapsto x_i$. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ has compact support. We claim that there exists $\mathcal{U} \subseteq \mathrm{Diff}_c(\mathbb{R}^n)$ such that if $f \in \mathcal{U}$, then for any $v \in \mathbb{R}^{n-1}$, $f_v : \mathbb{R} \to \mathbb{R}$ defined by $f_v(x) = \pi_n f(v, x)$ is a diffeomorphism. Note that if $\pi_n f(v, x_1) = \pi_n f(v, x_2)$, then by the Mean Value Theorem, $f_v(x_3) = 0$ at some $x_3 \in [x_1, x_2]$, and

$$\begin{aligned}\delta(\mathrm{id}, f_v) &\geq \left|\tfrac{d}{dx}\,\mathrm{id}(x) - \tfrac{d}{dx} f_v(x)\right|\Bigg|_{x=x_3} \\ &= |1 - 0| \\ &= 1.\end{aligned}$$

Note that $f \mapsto \sup_{v \in \mathbb{R}^{n-1}} \delta(\mathrm{id}, f_v)$ is continuous, so the preimage of $(-1, 1)$ under it is open. Let this preimage be $\mathcal{U}$.

Given some $f \in \mathrm{Diff}_c(\mathbb{R}^n)$ in $\mathcal{U}$, define $G_i(v, x) = \pi_i(v, f_v^{-1}(x))$, and define $H, G : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H(v, x) = (v, f_v(x))$$

and

$$G(v, x) = (G_1(v, x), \ldots, G_{n-1}(v, x), x).$$

Clearly H and G are smooth in terms of f, and $G_i \circ H = \pi_i \circ f$, so $G \circ H = f$. □

We now come to our direct application of Theorem 4.2.

Lemma 5.2. *Let $U, V \subseteq \mathbb{R}^n$ be open such that the closure of U is compact and contained in V. Then there exist vector fields $X_1, \ldots X_4 \in \mathfrak{X}_c(\mathbb{R}^n)$ whose supports are contained in V such that the following condition holds:*

- *There is an open* $\mathcal{U} \subseteq \mathrm{Diff}_c(\mathbb{R}^n)$ *containing* id *such that for every* $g \in \mathcal{U}$ *with* $\mathrm{supp}(g) \subseteq U$*, if* g *preserves vertical lines, then* g *can be decomposed as*
$$g = [G_1, \exp(Y_1)] \dots [G_4, \exp(Y_4)],$$
with each $G_i : \mathrm{Diff}_c(\mathbb{R}^n) \to \mathrm{Diff}_c(\mathbb{R}^n)$ *smooth.*

Proof. Let B^{n-1} be a ball in $\mathbb{R}^{n-1}$. Then S^1 *foliates* U inside of V, in that we may construct a smooth embedding $\phi : B^{n-1} \times S^1 \to \mathbb{R}^n$ with $U \subseteq \phi(\{b\} \times S^1) \subseteq V$ such that $\phi(S^1 \times \{b\})$ is a subset of a vertical line for every $b \in B^{n-1}$. One way to do this is to use the compactness of the closure of U, and the resulting boundedness of U, to take some cylinder $B^{n-1} \times I$ containing U for some open interval I, and from there embedding the rest of the circle. See [14] for a nice visualization of this.

If $g \in G_U$ preserves vertical lines, then we may instead consider it as a diffeomorphism $\mathbb{R}^n$ to $\mathbb{R}^n$ of the form $(v, y) \mapsto \hat{g}(v, y)$ for $v \in \mathbb{R}^{n-1}$ and some $\hat{g}$. Letting $g_v(y)$ denote $\hat{g}(v, y)$ and $\tilde{g}_v$ denote $(v, y) \mapsto (v, g_v(y))$, each $\tilde{g}_v$ has support on the vertical line $\{v\} \times \mathbb{R}$ and thus on $\{b_v\} \times S^1$ for some b_v. Letting $\sigma_v : S^1 \to B^{n-1} \times S^1$ be $s \mapsto (b_v, s)$, we can conjugate $\tilde{g}_v$ by $\phi \circ \sigma_b$, yielding a diffeomorphism of S^1. By Theorem 4.2, we may write this as
$$(\phi \circ \sigma_v)^{-1} \circ \tilde{g}_v \circ (\phi \circ \sigma_v) = [G_{v,1}, \exp(F_1)] \dots [G_{v,4}, \exp(F_4)],$$
with F_4 independent of $\tilde{g}_v$ and $G_{v,i}$ a smooth function of $\tilde{g}_v$. We conjugate the F_i's this time by the differential map $D_x(\phi \circ \sigma_v)^{-1} : T_x S^1 \to T_{(\phi\circ\sigma_v)^{-1}(x)}(\phi(\{b_v\} \times S^1))$, giving us vector fields $F_{v,i} = D_x(\phi \circ \sigma_v)^{-1} F_i$ on $(\phi \circ \sigma_v)(S^1)$. We also conjugate the $G_{v,i}$'s by $(\phi \circ \sigma_v)^{-1}$, yielding diffeomorphisms $G_{v,i'} = (\phi \circ \sigma_b)^{-1} \circ G_{v,i} \circ (\phi \circ \sigma_b)$ of $(\phi \circ \sigma_b)(\{b_v\} \times S^1)$. Because of the smooth dependence of $G_{v,i}$ on $\tilde{g}_v$, $G_{v,i'}$ is also a smooth function of $\tilde{g}_v$, so the $G_{v,i'}$'s piece together to form smooth functions $G_{i'}$ on $\phi(B^{n-1} \times S^1)$. Since $g(x) = \mathrm{id}(x)$ when $x \notin \phi(B^{n-1} \times S^1)$, we can trivially extend each $G_{i'}$ to a diffeomorphism on the entirety of $\mathbb{R}^n$. We can piece together the vector fields because of their smooth dependencies on v, and then we can extend these pieced together vector fields to smooth vector fields F_i on all of $\mathbb{R}^n$ supported on $V \supsetneq \phi(B^{n-1} \times S^1)$. Thus, we have the decomposition $g = [G_{1'}, \exp(F_{1'})] \dots [G_{4'}, \exp(F_{4'})]$, with all of the supports contained in V and each $G_{i'}$ a smooth function of g. □

With Lemma 5.1 and Lemma 5.2 in hand, we may proceed to the proof of our main theorem.

Proof of Theorem 2.6. We prove that $\mathrm{Diff}_c(\mathbb{R}^n)$ is perfect for $n \geq 2$ via induction on n, and then invoke Proposition 3.1 to yield the full theorem.

We begin with the base case $n = 2$. We show that for an arbitrary compact $K \subseteq \mathbb{R}^2$ which is the closure of an open set, every element of G_K is a product of a fixed number of commutators, which yields the theorem because definitionally every element of $\mathrm{Diff}_c(\mathbb{R}^2)$ is in some G_K. Take said arbitrary K, and take some open set U containing K such that the closure of U is compact. In turn, take some open set V containing the closure of U such that the closure of K is compact. Applying Lemma 5.2 yields a neighborhood $\mathcal{U}$ of id such that every $g \in \mathcal{U}$ with $\mathrm{supp}(g) \subseteq U$ preserving horizontal lines, g can be decomposed as $[G_1, \exp(Y_1)] \dots [G_4, \exp(Y_4)]$. In fact, we may symmetrically apply Lemma 5.2 to find a neighborhood $\mathcal{V}$ satisfying the same hypotheses, except we assume that g preserves horizontal lines, and we have a decomposition $[G_5, \exp(Y_5)] \dots [G_8, \exp(Y_8)]$. We invoke Lemma 5.1, yielding a decomposition $g = G(g) \circ H(g)$ for any g in some open $\mathcal{W}$. By smoothness of G and H, $\mathcal{T} = \mathcal{W} \cap G^{-1}(\mathcal{V}) \cap H^{-1}(\mathcal{U})$ is open. For any $g \in \mathcal{T}$, $G(g)$ and $H(g)$ are in $\mathcal{V}$ and $\mathcal{U}$ respectively, so

$$g = [G_5, \exp(Y_5)] \dots [G_8, \exp(Y_8)][G_1, \exp(Y_1)] \dots [G_4, \exp(Y_4)].$$

Thus, every element of $\mathcal{T}$ is a product of commutators. By Lemma 2.3, $\mathcal{T}$ generates G_U, so every element of G_U and thus $G_K \subseteq G_U$ is a product of commutators. Note that the number of commutators involved is fixed.

We now perform the inductive step, which is very similar to the base case. Assume that for $n \leq k$, there exists a natural $r(n)$ such that any $g \in \mathrm{Diff}_c(\mathbb{R}^n)$ can be decomposed into a product of commutators $g = [G_1, \exp(F_1)] \dots [G_{r(n)}, \exp(F_{r(n)})]$, with each G_i smooth and each F_i independent of g. Let K be compact and the closure of an open set, and let $U, V \subseteq \mathbb{R}^{k+1}$ be open such that K is contained in U and the closure of U is contained in V. Take the neighborhood $\mathcal{U}$ given to us by Lemma 5.2 and the neighborhood $\mathcal{V}$ given by Lemma 5.1. Taking the decomposition $G \circ H$ from Lemma 5.1, and letting f be an element of $\mathcal{V} \cap H^{-1}(\mathcal{U})$, the action of $G(f)$ on each horizontal hyperplane $\mathbb{R}^k \times \{x\}$ can be decomposed by the inductive hypothesis as $G_x(f) = [G_{x,1}, \exp(F_{x,1})] \dots [G_{x,r(n)}, \exp(F_{x,r(n)})]$. The proof of Lemma 5.2 lets us piece together the $G_{x,i}$'s and $F_{x,i}$'s into diffeomorphisms and vector fields on all of $\mathbb{R}^{k+1}$, so we have $G(f) = [G_1, \exp(F_1)] \dots [G_{r(n)}, \exp(F_{r(n)})]$. Therefore, Lemma 5.2 tells us that we may decompose $H(f)$ as

$$H(f) = [G_{r(n)+1}, \exp(F_{r(n+1)})] \dots [G_{r(n)+4}, \exp(F_{r(n+4)})].$$

We have

$$\begin{aligned} f =& [G_1, \exp(F_1)] \dots [G_{r(n)}, \exp(F_{r(n)})] \\ & \circ [G_{r(n)+1}, \exp(F_{r(n+1)})] \dots [G_{r(n)+4}, \exp(F_{r(n+4)})]. \end{aligned}$$

Thus, every element of $\mathcal{V} \cap H^{-1}(\mathcal{U})$ is a product of commutators, so by Lemma 2.3, every element of G_K is a product of commutators. Because K was arbitrary, we conclude the proof. $\square$

We recall, as an immediately accessible and centrally important application of Theorem 2.6, that if $\mathrm{Diff}_c(M)$ is perfect, then $\mathrm{Diff}_0(M)$ is simple (this is Lemma 2.4).

Corollary 5.3. *For every smooth manifold M, $\mathrm{Diff}_0(M)$ is simple.*

6. Conclusion

We give several resources with which the interested reader could study diffeomorphism groups further. First and foremost is Banyaga's *The Structure of Classical Diffeomorphism Groups* [2], the standard text on the topic, and a source of a great number of references for related topics such as the algebraic topology or infinite-dimensional Lie theory of diffeomorphism groups. Another expository work, which gives few proofs but a nice overview of the field, is [15].

We close with a short sketch of the history of the study of perfectness and simplicity of diffeomorphism groups, both to place the paper in context and to give candidates for further reading. The history begins not with diffeomorphisms themselves, though, but with homeomorphisms. In the 1960s and 1970s, the works of Anderson, Černavskiĭ, Edwards, and Kirby [1, 4, 12, 23] resulted in the theorem that the group of homeomorphisms isotopic to the identity of a manifold M is always simple. As a result, Smale conjectured that $\mathrm{Diff}^r_c(M)$, the group of C^r diffeomorphisms [18] with compact support and isotopic to the identity is simple. The first progress on Smale's conjecture was the work of Epstein, showing that the commutator subgroup of $\mathrm{Diff}^r_c(M)$ is simple. With the question of simplicity reduced to perfectness, Herman [10] showed that $\mathrm{Diff}^r_c(T^n)$, where T^n is the n-torus, is perfect using the Nash-Moser-Sergeraert Implicit Function Theorem. This was followed by the full result from Thurston [19], who used perfectness of $\mathrm{Diff}^r_c(T^n)$ to reduce to prove perfectness and thus simplicity of $\mathrm{Diff}^r_c(M)$. Thurston's proof utilizes a deep connection between the perfectness of the groups and the homology of certain classifying space of foliations.

Following Thurston, Mather [16, 17] provided another proof of simplicity of $\mathrm{Diff}^r_c(M)$, assuming that $1 \leq r \leq \infty$ and $r \neq \dim(M) + 1$, and Epstein improved Mather's methods in the case $r = \infty$ [5]. In more recent years, new proofs of perfectness and interest in commutator bounds have surfaced. For the latter, the number of commutators needed to express an element of $\mathrm{Diff}_c(\mathbb{R}^n)$, $\mathrm{Diff}_0(S^n)$, $\mathrm{Diff}_0(M)$ for compact, 3-dimensional M are uniformly bounded by 2, 4, and 10 have been proved by Burago et al. [3], with related uniform bounds [20] and non-uniform bounds [21, 22] from Tsuboi. As it turns out, the uniform commutator width bounds of [3] actually take as an input our non-uniformly-bounded result of perfectness. In the vein of new proofs of perfectness, Haller et. al [9, 8] provided a proof improving on [6], including a stronger result, that the commutators can be chosen to depend smoothly on the diffeomorphism. This was even further streamlined by Mann [14], whose proof we have followed.

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Proof School, 973 Mission St., San Francisco, California, 94103, USA
Email address: derekaoki@acm.org