About Infinite Algebraic Curves whose Indefinite Lengths Equal an Elliptic Arc

an English translation of

De Infinitis Curvis Algebraicis Quarum Longitudo Indefinita Arcui Elliptico Aequatur

By Leonhard Euler

Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg 11, 1830, pp.95-99

Translated by Derek R. Aoki[a]

Abstract

Euler continues his study of algebraic curves with equal arc length, a subject to which he returned several times. After a brief review, he introduces an infinite family of curves with the same indefinite length as a given ellipse. However, this first family is by his own admission poorly motivated, so he derives directly a different but related family of curves with the same indefinite length as a given ellipse.

[a]The author would like to thank Derek Wilairat and the anonymous reviewer for helpful comments.

Translator's Note

Euler's work relating to elliptic integrals spanned his career and fills two of the 29 mathematics volumes of the *Opera Omnia*. As noted in the editor's forward for these volumes [7], Euler returned several times to the study of algebraic curves with equal arc length. He presented work on this general topic in 1775 (E590), in 1776 (E638, E639, E645), and in 1781 (E780–E783), as well as leaving an undated manuscript (E817).

The article translated here, E780 [5], proves that for any ellipse there is an infinite family of curves which share the same indefinite length. Appearing in 1830, it was among the last of his journal papers to be published by the St. Petersburg Academy [6]. Note that we follow the corrected (and lightly reformatted) *Opera Omnia* printing rather than the original *Mémoires* printing.

Translation

1. I had proposed some years before two theorems which certainly seemed to me worthy of all attention, one of which stated that *there is absolutely no algebraic curve to be given whose indefinite length is equal to any logarithm* [b]; the other denied that *other than the circle circle no algebraic curve may be exhibited whose indefinite length is equal to that of some circular arc.* [c] Whether indeed other curved lines may be given whose rectifications are proper to themselves, so that no other algebraic curves agree with the same, is a question of greatest difficulty.

2. I have indeed invented some algebraic curves, whose indefinite lengths are equal to elliptic [d] and even parabolic [e] arcs, but in truth still nothing had allowed me to investigate such an algebraic curve whose rectification would agree with that of a hyperbola. Recently, however, I fell into such formulas which yield innumerable algebraic curves, whose entire lengths may be reduced to elliptic arcs, for which reason it seems worthwhile to have introduced these curves, for this argument is entirely new and has not been discussed clearly enough by anyone.

3. I considered naturally the curve, whose orthogonal coordinates x and y

[b] See Theorem 3 of E590 [1].

[c] See Theorem 2 of E590 [1]. However, Euler later disproved this result in E783 [4].

[d] See E639 [2].

[e] See E638 [3].

are expressed in these formulas

$$x = \frac{a\cos.(n+1)\varphi}{n+1} + \frac{b\cos.(n-1)\varphi}{n-1},$$
$$y = \frac{a\sin.(n+1)\varphi}{n+1} + \frac{b\sin.(n-1)\varphi}{n-1}.$$

Hence it will be

$$\frac{\partial x}{\partial\varphi} = -a\sin.(n+1)\varphi - b\sin.(n-1)\varphi,$$
$$\frac{\partial y}{\partial\varphi} = a\cos.(n+1)\varphi + b\cos.(n-1)\varphi.$$

Thus the element of the curve will be

$$\sqrt{\partial x^2 + \partial y^2} = \partial\varphi\sqrt{aa + bb + 2ab\cos.2\varphi},$$

this formula obviously involving the rectification of an ellipse. For if the coordinates are set on the ellipse

$$X = f\cos.\varphi \qquad \text{and} \qquad Y = g\sin.\varphi,$$

it will be

$$\sqrt{\partial X^2 + \partial Y^2} = \partial\varphi\sqrt{ff\sin.\varphi^2 + gg\cos.\varphi^2},$$

on account of which the formulas

$$\sin.\varphi^2 = \frac{1-\cos.2\varphi}{2} \qquad \text{and} \qquad \cos.\varphi^2 = \frac{1+\cos.2\varphi}{2}$$

change into

$$\partial\varphi\sqrt{\frac{ff+gg}{2} + \frac{gg-ff}{2}\cos.2\varphi},$$

where if we take $g = a + b$ and $f = a - b$, our formula itself results, so that the semiaxes of the ellipses having the same rectification are thus $a+b$ and $a-b$.

4. Since therefore in the element of the curve $\sqrt{\partial x^2 + \partial y^2}$ there is no number n and thus it is left entirely to our discretion, it is obvious that there may be exhibited innumerable algebraic curves, whose arcs are thus equal to the arcs of the given ellipse, each very different from each other, and even the orders of the algebraic curves will be considered to be very different for various values taken in place of n. However it does not follow, even though circles may be a species of ellipse, that like circles there may be other diverse curves of the same rectification assigned this way. For when a circle arises, if both semiaxes f and g are set equal, it is necessary that either a or b vanish. Taking however $b = 0$, it will be

$$x = \frac{a\cos.(n+1)\varphi}{n+1} \qquad \text{and} \qquad y = \frac{a\sin.(n+1)\varphi}{n+1}$$

and thus it will be $xx + yy = \frac{aa}{(n+1)^2}$; whatever n from before is accepted, a circle therefore always arises.

5. Since moreover I had only fallen by chance into these formulas, it will be entirely worth the effort to inquire into an analysis of this sort, which, having been proposed for an ellipse, leads by hand, by a direct route, to the formulas brought above in §3, which I undertake to solve in the following problem.

THE PROBLEM

6. Given an ellipse, whose orthogonal coordinates X and Y are defined by these formulas

$$X = 2f\cos.\theta \qquad \text{and} \qquad Y = 2g\sin.\theta,$$

find innumerable other algebraic curves which share a common rectification with this ellipse.

THE SOLUTION

If x and y are the coordinates of the sought curve, and since it must be that $\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2$, this condition will be fulfilled, if it might be assumed that

$$\partial x = \partial X\cos.\varphi + \partial Y\sin.\varphi,$$
$$\partial y = \partial X\sin.\varphi - \partial Y\cos.\varphi.$$

Now since these differential formulas must admit integration, let them be integrated as usual, and it will be revealed that

$$x = X\cos.\varphi + Y\sin.\varphi + \int\partial\varphi(X\sin.\varphi - Y\cos.\varphi),$$
$$y = X\sin.\varphi - Y\cos.\varphi - \int\partial\varphi(X\cos.\varphi + Y\sin.\varphi).$$

7. Because already we have $X = 2f\cos.\theta$ and $Y = 2g\sin.\theta$, we take the angle $\varphi = n\theta$, and it will be through the known reductions of angles

$$\begin{aligned}
X\sin.\varphi &= f\sin.(n+1)\theta + f\sin.(n-1)\theta,\\
X\cos.\varphi &= f\cos.(n+1)\theta + f\cos.(n-1)\theta,\\
Y\sin.\varphi &= -g\cos.(n+1)\theta + g\cos.(n-1)\theta,\\
Y\cos.\varphi &= g\sin.(n+1)\theta - g\sin.(n-1)\theta.
\end{aligned}$$

From these values it is now deduced that

$$\begin{aligned}
X\sin.\varphi - Y\cos.\varphi &= (f-g)\sin.(n+1)\theta + (f+g)\sin.(n-1)\theta,\\
X\cos.\varphi + Y\sin.\varphi &= (f-g)\cos.(n+1)\theta + (f+g)\cos.(n-1)\theta,
\end{aligned}$$

which carried through with $\partial\varphi = n\partial\theta$ and integrated, if for the sake of brevity it is put $f+g=b$ and $f-g=a$, give

$$\begin{aligned}
\int \partial\varphi(X\sin.\varphi - Y\cos.\varphi) &= -\frac{na\cos.(n+1)\theta}{n+1} - \frac{nb\cos.(n-1)\theta}{n-1},\\
\int \partial\varphi(X\cos.\varphi + Y\sin.\varphi) &= +\frac{na\sin.(n+1)\theta}{n+1} + \frac{nb\sin.(n-1)\theta}{n-1}.
\end{aligned}$$

8. If therefore the integrals are substituted for these values, our coordinates are

$$\begin{aligned}
x &= a\cos.(n+1)\theta + b\cos.(n-1)\theta - \frac{na}{n+1}\cos.(n+1)\theta - \frac{nb}{n-1}\cos.(n-1)\theta,\\
y &= a\sin.(n+1)\theta + b\sin.(n-1)\theta - \frac{na}{n+1}\sin.(n+1)\theta - \frac{nb}{n-1}\sin.(n-1)\theta.
\end{aligned}$$

But with the two members properly joined, these coordinates for the curves having a common rectification with the ellipse will be expressed as follows

$$\begin{aligned}
x &= \frac{a}{n+1}\cos.(n+1)\theta - \frac{b}{n-1}\cos.(n-1)\theta,\\
y &= \frac{a}{n+1}\sin.(n+1)\theta - \frac{b}{n-1}\sin.(n-1)\theta,
\end{aligned}$$

which do not differ from those expressions cited above, except that here the letter b may be taken negatively. Where in the known case, in which $n=0$, the same ellipse will result. For putting $n=0$, it will be

$$x = (a+b)\cos.\theta \qquad \text{and} \qquad y = (a-b)\sin.\theta.$$

9. If $n = 2$ is taken, without a doubt the simplest curve will result. Moreover it will be found

$$x = \frac{a}{3}\cos.3\theta - b\cos.\theta \qquad \text{and} \qquad y = \frac{a}{3}\sin.3\theta - b\sin.\theta.$$

In place of $\frac{a}{3}$ we write the letter c, and we seek the chord $\sqrt{xx+yy} = z$, and it will be $zz = cc + bb - 2bc\cos.2\theta$. Consequently,

$$\cos.2\theta = \frac{bb+cc-zz}{2bc}$$

and hence [f]

$$\sin.\theta = \sqrt{\frac{zz-(b-c)^2}{4bc}} \qquad \text{and} \qquad \cos.\theta = \sqrt{\frac{(b+c)^2-zz}{4bc}}.$$

Hence, because $\sin.3\theta = 4\sin.\theta\cos.\theta^2 - \sin.\theta$ and $\cos.3\theta = 4\cos.\theta^3 - 3\cos.\theta$, if the angle θ is eliminated, an equation between the same coordinates x and y will arise, which will however rise to more dimensions.[g]

10. The method by which we investigated these formulas is however much broader, and can be extended to take other curves in place of an ellipse. For if the coordinates for the curve have been given

$$X = 2f\cos.\alpha\theta + 2f'\cos.\beta\theta + \text{etc.},$$
$$Y = 2g\sin.\alpha\theta + 2g'\sin.\beta\theta + \text{etc.},$$

then for the rest of the curves have a common rectification with the proposed ones, by assuming again

$$f - g = a, \quad f + g = b, \qquad \text{and} \qquad f' - g' = a', \quad f' + g' = b' \qquad \text{etc.}$$

it will become

$$\begin{aligned} x &= \frac{\alpha a}{n+\alpha}\cos.(n+\alpha)\theta - \frac{ab}{n-\alpha}\cos.(n-\alpha)\theta \\ &\quad + \frac{\beta a'}{n+\beta}\cos.(n+\beta)\theta - \frac{\beta b'}{n-\beta}\cos.(n-\beta)\theta + \text{etc.}, \\ y &= \frac{\alpha a}{n+\alpha}\sin.(n+\alpha)\theta - \frac{ab}{n-\alpha}\sin.(n-\alpha)\theta \\ &\quad + \frac{\beta a'}{n+\beta}\sin.(n+\beta)\theta - \frac{\beta b'}{n-\beta}\sin.(n-\beta)\theta + \text{etc.} \end{aligned}$$

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[f] The editor, Krazer, silently corrects two misapplications of the double-angle identities for $\cos 2\theta$, writing $4bc$ in the denominators in place of the $2bc$ found in the *Mémoires*.

[g] Krazer follows the *Mémoires* in writing these triple angle identities, but in modern notation this would more unambiguously be written $\sin.3\theta = 4\sin.\theta(\cos.\theta)^2 - \sin.\theta$ and $\cos.3\theta = 4(\cos.\theta)^3 - 3\cos.\theta$

Where again because of the indefinite number n innumerable curves come about.

References

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De Infinitis Curvis Algebraicis Quarum Longitudo Indefinita Arcui Elliptico Aequatur

Auctore L. Eulero

Mémoires de l'Académie Impériale des Sciences de St. Pétersbourg 11, 1830, pp.95-99

1. Proposueram ante aliquot annos duo theoremata, quae mihi quidem omni attentione digna videbantur quorum altero statui *nullam prorsus dari curvam algebraicam, cuius longitudo indefinita cuipiam logarithmo aequatur*; altero vero negavi *praeter circulum ullam exhiberi posse curvam algebraicam, cuius longitudo indefinita arcui cuipiam circulari aequatur.* Utrum vero aliae dentur lineae curvae, quarum rectificatio ita ipsis sit propria, ut eadem nullis aliis curvis algebraicis conveniat, quaestio est maxime ardua.

2. Inveni quidem nonnullas curvas algebraicas, quarum longitudo indefinita aequatur arcui elliptico atque adeo etiam parabolico, at vero nullam adhuc investigare mihi licuit eiusmodi curvam algebraicam, cuius rectificatio cum hyperbola conveniret. Nuper autem incidi in eiusmodi formulas, quae infinitas praebent curvas algebraicas, quarum omnium longitudo indefinita ad arcum ellipticum reduci potest, quas idcirco curvas hic in medium attulisse operae pretium videtur, siquidem hoc argumentum plane est novum neque a quoquam satis dilucide pertractatum.

3. Consideravi scilicet curvam, cuius coordinatae orthogonales x et y his formulis exprimantur

$$x = \frac{a\cos.(n+1)\varphi}{n+1} + \frac{b\cos.(n-1)\varphi}{n-1},$$
$$y = \frac{a\sin.(n+1)\varphi}{n+1} + \frac{b\sin.(n-1)\varphi}{n-1}.$$

Hinc ergo erit

$$\frac{\partial x}{\partial \varphi} = -a\sin.(n+1)\varphi - b\sin.(n-1)\varphi,$$
$$\frac{\partial y}{\partial \varphi} = a\cos.(n+1)\varphi + b\cos.(n-1)\varphi.$$

Hinc ergo erit elementum curvae

$$\sqrt{\partial x^2 + \partial y^2} = \partial\varphi\sqrt{aa + bb + 2ab\cos.2\varphi},$$

quae formula manifesto rectificationem ellipsis involvit. Nam si coordinatae statuantur in ellipsi

$$X = f\cos.\varphi \qquad \text{et} \qquad Y = g\sin.\varphi,$$

erit

$$\sqrt{\partial X^2 + \partial Y^2} = \partial\varphi\sqrt{ff\sin.\varphi^2 + gg\cos.\varphi^2},$$

quae formula ob

$$\sin.\varphi^2 = \frac{1-\cos.2\varphi}{2} \qquad \text{et} \qquad \cos.\varphi^2 = \frac{1+\cos.2\varphi}{2}$$

abit in hanc

$$\partial\varphi\sqrt{\frac{ff+gg}{2}+\frac{gg-ff}{2}\cos.2\varphi},$$

ubi si sumamus $g = a + b$ et $f = a - b$, ipsa nostra formula resultat, ita ut ellipseos eandem rectificationem habentis sint semiaxes $a + b$ et $a - b$.

4. Quoniam igitur in elemento curvae $\sqrt{\partial x^2 + \partial y^2}$ numerus n non inest ideoque arbitrio nostro prorsus relinquitur, manifestum est innumerabiles exhiberi posse curvas algebraicas, quarum arcus adeo datae ellipseos arcubus aequentur, quae omnes curvae inter se maxime erunt diversae atque pro variis valoribus loco n assumtis ad ordines curvarum algebraicarum plurimum diversos erunt referendae. Neque tamen hinc sequitur, etiamsi circulus sit species ellipsis, pro circulo quoque alias diversas curvas eiusdem rectificationis hoc modo assignari posse. Cum enim circulus prodeat, si ambo semiaxes f et g statuantur aequales, necesse est, ut vel a vel b evanescat. Sumto autem $b = 0$ erit

$$x = \frac{a\cos.(n+1)\varphi}{n+1} \qquad \text{et} \qquad y = \frac{a\sin.(n+1)\varphi}{n+1}$$

sicque erit $xx + yy = \frac{aa}{(n+1)^2}$; quicquid pro n accipiatur, semper igitur circulus oritur.

5. Cum autem casu in istas formulas tantum incidissem, utique operae pretium erit in eiusmodi Analysin inquirere, quae proposita ellipsi via directa ad formulas supra § 3 allatas manuducat, quem in finem sequens problema resolvendum suscipio.

PROBLEMA.

6. Proposita ellipsi, cuius coordinatae orthogonales X et Y his formulis definiantur

$$X = 2f\cos.\theta \qquad \text{et} \qquad Y = 2g\sin.\theta,$$

invenire innumerabiles alias curvas algebraicas, quae cum ista ellipsi communem rectificationem sortiantur.

SOLUTIO.

Sint x et y coordinatae curvarum quaesitarum, et cum esse oporteat $\partial x^2 + \partial y^2 = \partial X^2 + \partial Y^2$, haec conditio implebitur, si sumatur

$$\partial x = \partial X \cos.\varphi + \partial Y \sin.\varphi,$$
$$\partial y = \partial X \sin.\varphi - \partial Y \cos.\varphi.$$

Iam quia hae formulae differentiales integrationem admittere debent, integrentur, qua fieri licet, more solito ac reperietur

$$x = X \cos.\varphi + Y \sin.\varphi + \int \partial\varphi(X \sin.\varphi - Y \cos.\varphi),$$
$$y = X \sin.\varphi - Y \cos.\varphi - \int \partial\varphi(X \cos.\varphi + Y \sin.\varphi).$$

7. Cum iam sit $X = 2f \cos.\theta$ et $Y = 2g \sin.\theta$, sumamus angulum $\varphi = n\theta$ eritque per notas angulorum reductiones

$$X \sin.\varphi = f \sin.(n+1)\theta + f \sin.(n-1)\theta,$$
$$X \cos.\varphi = f \cos.(n+1)\theta + f \cos.(n-1)\theta,$$
$$Y \sin.\varphi = -g \cos.(n+1)\theta + g \cos.(n-1)\theta,$$
$$Y \cos.\varphi = g \sin.(n+1)\theta - g \sin.(n-1)\theta.$$

Ex his iam valoribus colligitur

$$X \sin.\varphi - Y \cos.\varphi = (f-g)\sin.(n+1)\theta + (f+g)\sin.(n-1)\theta,$$
$$X \cos.\varphi + Y \sin.\varphi = (f-g)\cos.(n+1)\theta + (f+g)\cos.(n-1)\theta,$$

quae aequationes ductae in $\partial\varphi = n\partial\theta$ et integratae, si brevitatis gratia ponatur $f+g=b$ et $f-g=a$, dabunt

$$\int \partial\varphi(X \sin.\varphi - Y \cos.\varphi) = -\frac{na\cos.(n+1)\theta}{n+1} - \frac{nb\cos.(n-1)\theta}{n-1},$$
$$\int \partial\varphi(X \cos.\varphi + Y \sin.\varphi) = +\frac{na\sin.(n+1)\theta}{n+1} + \frac{nb\sin.(n-1)\theta}{n-1}.$$

8. Si igitur pro integralibus hi valores substituantur, nostrae coordinatae erunt

$$x = a\cos.(n+1)\theta + b\cos.(n-1)\theta - \frac{na}{n+1}\cos.(n+1)\theta - \frac{nb}{n-1}\cos.(n-1)\theta,$$
$$y = a\sin.(n+1)\theta + b\sin.(n-1)\theta - \frac{na}{n+1}\sin.(n+1)\theta - \frac{nb}{n-1}\sin.(n-1)\theta.$$

At binis membris rite coniunctis istae coordinatae pro curvis quaesitis cum ellipsi communem rectificationem habentibus ita erunt expressae

$$x = \frac{a}{n+1}\cos.(n+1)\theta - \frac{b}{n-1}\cos.(n-1)\theta,$$
$$y = \frac{a}{n+1}\sin.(n+1)\theta - \frac{b}{n-1}\sin.(n-1)\theta,$$

quae expressiones a supra allatis aliter non differunt, nisi quod hic littera b negative sit sumta. Ubi notandum casu, quo $n = 0$, ipsam ellipsin esse prodituram. Posito enim $n = 0$ fiet

$$x = (a+b)\cos.\theta \qquad \text{et} \qquad y = (a-b)\sin.\theta.$$

9. Si sumatur $n = 2$, prodibit sinpe dubio curva post ellipsin simplicissima. Reperietur autem

$$x = \frac{a}{3}\cos.3\theta - b\cos.\theta \qquad \text{et} \qquad y = \frac{a}{3}\sin.3\theta - b\sin.\theta.$$

Loco $\frac{a}{3}$ scribamus litteram c et quaeramus chordam $\sqrt{xx+yy} = z$ eritque $zz = cc + bb - 2bc\cos.2\theta$, consequenter

$$\cos.2\theta = \frac{bb+cc-zz}{2bc}$$

hincque

$$\sin.\theta = \sqrt{\frac{zz-(b-c)^2}{4bc}} \qquad \text{et} \qquad \cos.\theta = \sqrt{\frac{(b+c)^2-zz}{4bc}}.$$

Hinc, cum sit $\sin.3\theta = 4\sin.\theta\cos.\theta^2 - \sin.\theta$ et $\cos.3\theta = 4\cos.\theta^3 - 3\cos.\theta$, si angulus θ eliminetur, eruetur aequatio inter ipsas coordinatas x et y, quae autem ad plures dimensiones assurget.

10. Methodus, qua has formulas indagavimus, etiam multo latius patet atque ad alias curvas loco ellipsis assumtas extendi poterit. Si enim coordinatae pro curva data fuerint

$$X = 2f\cos.\alpha\theta + 2f'\cos.\beta\theta + \text{etc.},$$
$$Y = 2g\sin.\alpha\theta + 2g'\sin.\beta\theta + \text{etc.},$$

pro reliquis curvis cum proposita communem rectificationem habentibus ponendo iterum

$$f - g = a, \quad f + g = b, \qquad \text{et} \qquad f' - g' = a', \quad f' + g' = b' \qquad \text{etc.}$$

fiet

$$\begin{aligned} x = {} & \frac{\alpha a}{n+\alpha}\cos.(n+\alpha)\theta - \frac{ab}{n-\alpha}\cos.(n-\alpha)\theta \\ & + \frac{\beta a'}{n+\beta}\cos.(n+\beta)\theta - \frac{\beta b'}{n-\beta}\cos.(n-\beta)\theta + \text{etc.}, \\ y = {} & \frac{\alpha a}{n+\alpha}\sin.(n+\alpha)\theta - \frac{ab}{n-\alpha}\sin.(n-\alpha)\theta \\ & + \frac{\beta a'}{n+\beta}\sin.(n+\beta)\theta - \frac{\beta b'}{n-\beta}\sin.(n-\beta)\theta + \text{etc.} \end{aligned}$$

Ubi iterum ob n numerum indefinitum innumerabiles curvae prodeunt.